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Nonlinear Best Approximations on Discrete Sets

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1. INTRODUCTION

In this paper, the existence of a best approximation to a continuous function f over a discrete set contained in $[0, 1]$ with respect to certain non-linear approximating families from $C[0, 1]$ is shown when f is a "normal point" and when the discrete set is sufficiently fine. This is accomplished by showing that the Remes algorithm can be used to generate a sequence of approximating functions which converges to a function which turns out to be a best approximation over the discrete set. This also shows that a best approximation over a discrete set can be found by using the Remes algorithm. In addition, we generalize the result, due to Werner [7], that as the discrete set fills up the interval, best approximation over the discrete set converges to best approximation over the interval. It is essential that this occur if one is to try to compute a best approximation over $[0, 1]$ using a computer since the computer uses only a finite number of points in $[0, 1]$.

2. DESCRIPTION OF APPROXIMATING FAMILY

Let P be an open set in Euclidean N-space. Consider the family V of real valued functions $F(A, x)$, where $A = (a_1, \ldots, a_N) \in P$ and $x \in [0, 1]$. The functions $F(A, x)$ and $\partial F(A, x)/\partial a_i$, $i = 1, 2, \ldots, N$, are assumed to be continuous in both A and x. In addition, the following conditions must be satisfied.

I. For each $A \in P$, the functions $\partial F(A, x)/\partial a_i$, $i = 1, \ldots, N$ generate a Haar subspace of dimension $d(A) \geqslant 1$. (If $h(x)$ is any nonzero element of the span of $\{\partial F(A, x)/\partial a_i\}_{i=1}^{N}$, then $h(x)$ has at most $d(A) - 1$ distinct zeros.)

II. For each $A \in P$, $F(A, x) \neq F(A', x)$ implies that $F(A, x) - F(A', x)$ has at most $d(A) - 1$ zeros. Among the nonlinear families that can be treated under the theory are the family of rational functions

$R_m{}^n = \{p/q\colon \partial p \leqslant n,\ \partial p \leqslant m,\ q > 0 \text{ on } [0, 1],\ p \text{ and } q \text{ are relative prime}\}$, where p and q are polynomials with ∂p denoting the degree of p and the family of exponential functions

$$G = \left\{\sum a_i e^{a_{n+i}x} : a_{n+i} \neq a_{n+j}, \quad \text{for} \quad i \neq j, \quad i, j = 1,..., n\right\}.$$

For more information on these and other families that meet the above conditions, see [2].

3. Preliminary Results and Notation

$F(A^*)$ is best approximation to $f \in C[0, 1]$ over $[0, 1]$ if $\| F(A^*) - f\| \leqslant \| F(A) - f\|$ for all $A \in P$, where $\| f \| = \max_{0 \leqslant x \leqslant 1} | f(x)|$. If M denotes a finite subset of $[0, 1]$, then $F(A^M)$ is a best approximation to f over M if $\| F(A^M) - f\|_M \leqslant \| F(A) - f\|_M$ for all $A \in P$, where $\| f\|_M = \max_{x \in M} | f(x)|$.

Definition 1. If n denotes the maximal value of $d(A)$ for $A \in P$, a function $f \in C[0, 1]$ is called a normal point if it has a best approximation $F(A^*)$ over $[0, 1]$ with $d(A^*) = n$.

If f is a normal point and $F(A^*)$ is a best approximation to f over $[0, 1]$, then by the characterization theorem due to Meinardus [5], there exist a set of $n + 1$ critical points $0 \leqslant x_1 < x_2 < \cdots < x_{n+1} \leqslant 1$ such that

$$F(A^*, x_i) - f(x_i) = -(F(A^*, x_{i+1}) - f(x_{i+1})) \tag{1}$$

for $i = 1,..., n$ and

$$\| F(A^*) - f\| = | F(A^*, x_i) - f(x_i)| \tag{2}$$

for $i = 1,..., n + 1$. Since the critical points do not have to be unique, we make the following definition.

Definition 2. Let $D = \{u = (u_1,..., u_{n+1}) \in E_{n+1} : 0 \leqslant u_1 < \cdots < u_{n+1} \leqslant 1\}$ and let $C = \{u \in D$: (1) and (2) hold with x_i replaced by $u_i\}$. Also, let $\| u \| = \max_{1 \leqslant i \leqslant n+1} | u_i |$.

In Barrar and Loeb [1], we find the following result.

Lemma 1. *Let* $\bar{A} = (\bar{a}_1, \bar{a}_2,..., \bar{a}_N) \in P$ *and* $d(A) = q$. *Further, let* $x_1, x_2,..., x_q$ *be distinct points in* $[0, 1]$ *and set* $F(\bar{A}, x_i) = c_i$ *for* $i = 1, 2,..., q$. *Then, for sufficiently small* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that the equations*

$$F(A, x_i) = \hat{c}_i, \qquad i = 1, 2,..., q, \tag{3}$$

with $|c_i - \hat{c}_i| \leqslant \delta$ *having a unique solution* $A = (a_1, a_2, ..., a_N)$ *such that* $a_i = \bar{a}_i$ *for* $q + 1 \leqslant i \leqslant N$ *and* $\| A - \bar{A} \| \leqslant \epsilon$[1] (i.e., $F(A, x)$ *satisfies* (3) *implies that an* A' *equivalent to* A *can be found such that* $\| A' - A \| \leqslant \epsilon$).

The proof of the next result is due to Barrar and Loeb [1].

LEMMA 2. *Let* f *be a normal point with best approximation* $F(A^*)$. *There exists a* $\delta^* > 0$ *such that for* $\delta \leqslant \delta^*$ $L = \{F(A) \in V: \| F(A^*) - F(A)\| \leqslant \delta\}$ *is compact.*

Proof. Let $x_1, ..., x_n$ be n distinct points in [0, 1]. Choose $\epsilon_1 > 0$ such that $\| A - A^* \| \leqslant \epsilon_1$ implies that $A \in P$., Now, choose $\epsilon > 0$ and $\delta > 0$ as in Lemma 1 such that $\epsilon < \epsilon_1$. Let $\delta^* = \delta$ and let $\{F(A)^k\}_{k=1}^{\infty}$ be a sequence in L. Then, $| F(A^k, x_i) - F(A^*, x_i)| < \delta^*$. By Lemma 1, there exists $\bar{A}^k$ equivalent to A^k, $k = 1, 2, ...$, such that $\| \bar{A}^k - A \| \leqslant \epsilon$. By going to subsequences if necessary, we can assume that $\lim_{k\to\infty} \bar{A}^k = \bar{A}$. Since $\| \bar{A} - A^* \| \leqslant \epsilon$, we have that $\bar{A} \in P$. Since $F(A, x)$ is continuous in A, we have that $\| F(\bar{A}) - F(A^*)\| = \lim_{k\to\infty} \| F(\bar{A}^k - F(A^*)\| \leqslant \delta^*$. Therefore, $F(\bar{A}) \in L$ and the conclusion holds.

Remark 1. By Ascoli's Theorem the set L is also equicontinuous.

LEMMA 3. *If* $F(A^*)$ *is a normal point and* $x_1, ..., x_n$ *are any* $n = d(A^*)$ *distinct points in* [0, 1], *then there exists a* $\delta > 0$, *such that* $| F(A^*, x_i) - \eta_i | \leqslant \delta$ *implies there is a unique* $F(A) \in V$ *such that* $F(A, x_i) = \eta_i$ *for* $i = 1, ..., n$ *and* $F(A) \in L$.

Proof. Choose $\epsilon > 0$ such that $\| A^* - A \| < \epsilon$ implies $F(A) \in L$. The conclusion follows immediately from Lemma 1.

From Barrar and Loeb [3] we have Lemmas 4 and 5.

LEMMA 4. *For arbitrary* $\epsilon > 0$, *there is a* $\delta > 0$ *such that for* $\hat{\delta} \leqslant \delta$ *we have*

(a) $D(\hat{\delta}) = \{u \in D: \text{dist}(u, C) \leqslant \hat{\delta}\}$ *is closed, and*

(b) $u \in D(\hat{\delta})$ *implies the system of equations*

$$F(A, u_i) - f(u_i) = (-1)^i \eta, \qquad \textit{for} \quad i = 1, ..., n + 1,$$

has a unique continuous solution $(A(u), \eta(u))$, *where* $A(u) \in P$ *and* $\| A - A^* \| < \epsilon$.

LEMMA 5. *There exists positive numbers* ϵ *and* δ *such that for any* $u \in D(\delta)$ *and* $A \in P$ *with the properties*

(a) $\| A - A^* \| < \epsilon$, *and*

[1] $\| A - \bar{A} \| = \max_{1 \leqslant i \leqslant N} | a_i - \bar{a}_i |$.

(b) $\operatorname{sign}(F(A, u_i) - f(u_i)) = -\operatorname{sign}(F(A, u_{i+1}) - f(u_{i+1})$ *for* $i = 1,..., n$, *the solution* $(A(u), \eta(u))$ *to the equations* $F(A, u_i) - f(u_i) = (-1)^i\eta$ *for* $i = 1,..., n+1$ *satisfy*

$$|\eta(u)| = \sum_{i=1}^{n+1} \theta_i \, |F(A, u_i) - f(u_i)|,$$

where $\sum_{i=1}^{n+1} \theta_i = 1$, $\theta_i \geqslant 1 - \theta$ *for* $i = 1,..., n+1$ *and* $\theta \in (0, 1)$.

We need one more result to ensure that the Remes algorithm can be carried out successfully.

LEMMA 6. *Assume that* $F(A^*)$ *is a best approximation over* [0, 1] *to the normal* $f \in C[0, 1]$ *with* $d(A^*) = n$. *Given that* $\delta > 0$, *there exists a number* K *where* $0 < K < \| f - F(A^*)\|$ *such that if*

(1) $y \in D$,

(2) $F(A) - f$ *alternates over* y, *and*

(3) $|F(A, y_i) - f(y_i)| \geqslant K$ *for* $i = 1,..., n+1$, *then* $y \in D(\delta)$.

Proof. The result is obvious if $f = F(A^*)$, so we assume that $\| F(A^*) - f\| > 0$. Now, assume that the result is false. Then, there exist sequences $\{K^m\}$, $\{A^m\}$, and $\{y^m\}$ such that $\{K^m\}$ converges to $\| F(A^*) - f\|$, $F(A^m) - f$ alternates over $y^m \in D$, $|F(A^m, y_i^m) - f(y_i^m)| \geqslant K^m$ for all m and i, and $y^m \notin D(\delta)$. By going to subsequences if necessary, we can assume that y^m converges to $y = (y_1, ..., y_{n+1}) \in D$; $y \notin C$ since none of the y^m are in $D(\delta)$. Without loss of generality, assume that $f(y_1^m) - F(A^m, y_1) > 0$ for all m. Let $F_i = \lim_{m\to\infty} \sup f(y_i^m) - F(A^m, y_i)$, for $i = 1,..., n+1$. Note that $F_i(-1)^{i+1} > 0$ and that $|F_i| = \infty$ is allowed. Also,

$$|F_i| \geqslant \lim_{m\to\infty} K^m = \| F(A^*) - f\|. \tag{4}$$

Claim. $f(y_i) - F(A^*, y_i) = F_i$

Proof of Claim. Assume false. Then, $|F_i| > |F(A^*, y_i) - f(y_i)|$ for some i. With no loss of generality, assume that $i = n+1$ and let $\gamma = |F_{n+1}| - |F(A^*, y_{n+1})|$. By Lemma 3, for sufficiently small ϵ, $\hat{\delta} > 0$ with $\epsilon < \gamma$, we can find a $A \in P$ such that

$$|F(A, y_i) - f(y_i)| \leqslant \| F(A^*) - f\| - \hat{\delta}, \qquad \text{for} \quad i = 1,..., n,$$

with $\| F(A^*) - F(A)\| < \epsilon/2$. Now, choose m so that

$$K^m > \| F(A^*) - f\| - \hat{\delta}/2$$

$$|F(A, y_i^m) - F(A, y_i)| < \hat{\delta}/4, \qquad \text{for} \quad i = 1,..., n,$$

$$|f(y_i^m) - f(y_i)| < \hat{\delta}/4, \qquad \text{for} \quad i = 1,..., n,$$

$$|F(A^*, y_{n+1}^m) - f(y_{n+1}^m) - [F(A^*, y_{n+1}) - f(y_{n+1})]| < \epsilon/2,$$

and

$$| F(A^m, y^m_{n+1}) - f(y^m_{n+1})| > | F(A^*, y_{n+1}) - f(y_{n+1})| + \epsilon.$$

Then, for $i = 1,..., n$

$$\begin{aligned} | F(A, y_i^m) - f(y_i^m)| &< | F(A, y_i) - f(y_i)| + \delta/2, \\ &\leqslant \| F(A^*) - f\| - \delta + \delta/2 < K^m, \\ &\leqslant | F(A^m, y_i^m) - f(y_i^m)|. \end{aligned}$$

Also,

$$\begin{aligned} | F(A, y^m_{n+1}) - f(y^m_{n+1})| &< | F(A^*, y^m_{n+1}) - f(y^m_{n+1})| + \epsilon/2, \\ &\qquad \text{since} \quad \| F(A^*) - F(A)\| < \epsilon/2, \\ &\leqslant | f(A^*, y_{n+1}) - F(y_{n+1})| + \epsilon, \\ &< | F(A^m, y^m_{n+1}) - f(y^m_{n+1})|. \end{aligned}$$

Thus,

$$| F(A^m, y_i^m) - f(y_i^m)| > | F(A, y_i^m) - f(y_i^m)|, \qquad \text{for} \quad i = 1,..., n + 1.$$

Since $F(A^m, y_i^m) - f$ alternates over $n + 1$ points, we must have that $F(A^m) - F(A)$ has n zeros, which is impossible. Thus, the claim follows.

By the claim, we have

$$| F_i | = | F(A^*, y_i) - f(y_i)| \leqslant \| F(A^*) - f\|. \tag{5}$$

Thus, from (4) and (5),

$$| F_i | = \| F(A^*) - f\|, \qquad \text{for all} \quad i.$$

But F_i is alternately positive and negative, hence, $y \in C$, which is a contradiction. Thus, the lemma holds.

4. Convergence of the Remes Algorithm

The Remes algorithm is described as follows:

ALGORITHM. *Having chosen A^m and u^m such that $u_i^m \in M$ for $i = 1,...,$ $n + 1$ and*

$$\operatorname{sign}(F(A^m, u_i^m) - f(u_i^m)) = - \operatorname{sign}(F(A^m, u^m_{i+1}) - f(u^m_{i+1})),$$

for $i = 1,..., n$, solve the system

$$F(A, u_i^m) - f(u_i^m) = (-1)^i \eta, \qquad \textit{for} \quad i = 1,..., n + 1$$

to obtain A^{m+1} and η^{m+1}. Then, choose u^{m+1} such that $u_i^{m+1} \in M$ for $i = 1,\ldots, n+1$ and

$$\text{(a)} \quad \operatorname{sign}(F(A^{m+1}, u_i^{m+1}) - f(u_i^{m+1})) = -\operatorname{sign}(F(A^{m+1}, u_{i+1}^{m+1}) - f(u_{i+1}^{m+1})),$$

for $i = 1,\ldots, n$

$$\text{(b)} \quad |\eta^{n+1}| \leqslant \min_{1 \leqslant i \leqslant n+1} |F(A^{m+1}, u_i^{m+1}) - f(u_i^{m+1})|,$$

and

$$\text{(c)} \quad \| F(A^{m+1}) - f \|_M = \max_{1 \leqslant i \leqslant n+1} |F(A^{m+1}, u_i^{m+1}) - f(u_i^{n+1})|.$$

To ensure that the algorithm can be carried out successfully, we define the following numbers.

(i) Choose $\delta^* > 0$ such that the set L in Lemma 2 is compact.

(ii) Choose ϵ, $\delta > 0$ such that Lemmas 4 and 5 are satisfied and such that $\| A - A^* \| < \epsilon$ implies $F(A) \in L$.

(iii) Choose K such that Lemma 6 is satisfied for the δ chosen in (ii).

We assume that the initial estimates A^0 and u^0 satisfy:

(a) $\| A^0 - A^* \| < \epsilon$,
(b) $F(A^0, x) - f(x)$ alternates in sign over u^0, and
(c) $\min_{1 \leqslant i \leqslant n+1} |F(A^0, u_i^0) - f(u_i^0)|, \geqslant K$.

THEOREM 1. *Let A^m denote the* mth *iterate of the Remes algorithm and let u^m denote the critical set associated with A^m. Then,*

(1) $\| A^m - A^* \| \leqslant \epsilon$,
(2) $u^m \in D(\delta)$,
(3) *$F(A^m, x) - f(x)$ alternates in sign over u_m, and*
(4) $\min_{1 \leqslant i \leqslant n+1} |F(A^m, u_i^m) - f(u_i^m)| \geqslant K$,

hold for $m = 0, 1, 2,\ldots,$.

Proof. The proof is by induction on m. For $m = 0$, we have (1), (3), and (4) by assumption. By Lemma 6, $u^0 \in D(\delta)$.

Now, assume that (1)–(4) hold for $m \leqslant k$. We solve the system $F(A, u_i^k) - f(u_i^k) = (-1)^i \eta$ to obtain A^{k+1} and η^{k+1}. The solution exists by Lemma 4 since $u^k \in D(\delta)$. Also by Lemma 4, $\| A^{k+1} - A^* \| < \epsilon$. Thus, (1) is satisfied for $m = k + 1$. By Lemma 5, $\eta^{k+1} = \sum_{i=1}^{n+1} \theta_i |F(A^k, u_i^k) - f(u_i^k)|$, where

$\sum_{i=1}^{n+1} \theta_i = 1$ and $\theta_i > 0$ for all i. Now, u^{k+1} is chosen such that $F(A^{k+1}, x) - f(x)$ alternates over u^{k+1}. Thus, (3) is satisfied. Also,

$$\begin{aligned}\min_{1 \leqslant i \leqslant n+1} |F(A^{k+1}, u_i^{k+1}) - f(u_i^{k+1})| &\geqslant \eta^{k+1}, \\ &\geqslant \sum_{i+1}^{n+1} \theta_i \min_{1 \leqslant i \leqslant n+1} |F(A^k, u_i^k) - f(u_i^k)|, \\ &= \min_{1 \leqslant i \leqslant n+1} |F(A^k, u_i^k)| \geqslant K.\end{aligned}$$

Therefore, (4) is satisfied. By Lemma 6, $u^{k+1} \in D(\delta)$. Thus, (2) is satisfied for $m = k + 1$ and the proof is complete.

We have shown that the algorithm can be carried out successfully. Now, we show that the sequence of functions generated by the algorithm converge to $F(A^M) \in V$ and that $F(A^M)$ is a best approximation to f over M.

THEOREM 2. *There is a $\delta > 0$ such that a best approximation to f over the finite set M exists whenever M is finer than δ and the sequence of functions generated by the Remes algorithm converge to $F(A^M)$.*

Proof. Choose δ, δ^*, ϵ, and K as above. From the continuity hypothesis of the nonlinear family, we certainly can find A^0 and u^0 with $u_i \in M$ that satisfies (a), (b), and (c) above by taking δ and ϵ even smaller. Then, the previous theorem guarantees a sequence $\{F(A^m)\}_{m=1}^{\infty}$ of well-defined functions in V. The set of all $u \in D(\delta)$ with $u_i \in M$ is finite. Hence, the sequence $\{F(A^m)\}_{m=1}^{\infty}$ has a finite number of distinct elements. From the proof of Theorem 1 and (b) in the Remes algorithm, we have

$$\eta^{k+1} \geqslant \min_{1 \leqslant i \leqslant n+1} |F(A^k, u_i^k) - f(u_i^k)| \geqslant \eta^k.$$

Thus, the sequence $\{\eta^m\}_{m=1}^{\infty}$ is monotone increasing and eventually must be constant. Thus, $\{F(A^m)\}_{m=1}^{\infty}$ eventually must be constant. Let $F(A^m) = \lim_{m \to \infty} F(A^m)$. Assume that $F(A^M)$ is not the best approximation to f over M. Then, there exists an $A \in P$ such that

$$\| F(A^M) - f \|_M > \| F(A) - f \|_M .$$

But there exist $n + 1$ points $x_1 < x_2 < \cdots < x_{n+1}$ such that

$$\| F(A^M) - f \|_M = | F(A^M, x_i) - f(x_i)|, \qquad \text{for} \quad i = 1, \ldots, n + 1$$

and $F(A^M) - f$ alternates over those points. This implies that $F(A^M) - F(A)$ has at least n distinct zeros, which is a contradiction. Thus, $F(A^M)$ is the best approximation to f over M.

5. On Convergence to $F(A^*)$

We have seen that if M is fine enough, then a normal point f has a best approximation over M. In this section, it is shown that as M becomes finer the best approximation to f over M converges to a best approximation over the interval. The result presented here is a generalization of one obtained by Werner [7], who was concerned only with the family of rational functions. We need the following result from [1], which is a generalizationof the familiar Strong Uniqueness Theorem.

THEOREM 3. *Let $F(A^*)$ be a best approximation to $f \in C[0, 1]$. There exists an $\alpha > 0$ such that for each $A \in P$*

$$\| F(A) - f \| \geqslant F(A^*) - f \| + \alpha \| F(A) - F(A^*) \|.$$

To obtain a measure of the rate of convergence, we use the following notation. If g is a continuous function on [0, 1] *let*

$$\omega_g(h) = \max_{|x-y| \leqslant h} | g(x) - g(y) |$$

and if $\mathscr{F}$ is a family of continuous functions on [0, 1] *let*

$$\Omega_{\mathscr{F}}(h) = \sup_{f \in \mathscr{F}} \omega_f(h).$$

$\Omega_{\mathscr{F}}$ is the joint modulus of continuity of the family $\mathscr{F}$.

THEOREM 4. *Let $f(x)$ be a normal point with modulus of continuity $\omega(h)$ and best approximation $F(A^*)$ over* [0, 1]. *For δ^*, δ chosen as in Lemma* 2 *and Theorem* 2, *respectively, let $\Omega(h)$ be the joint modulus of continuity for $L = \{F(A): \| F(A) - F(A^*) \| \leqslant \delta^*\}$. Then, for each grid M finer than δ, a best approximation $F(A^M)$ over M exists and satisfies*

$$\| F(A^M) - F(A^*) \| \leqslant K(\omega(\delta) + \Omega(\delta)),$$

where K depends only on f.

Proof. A best approximation $F(A^M)$ exists by Theorem 2. From Theorem 3 we have

$$\alpha \| F(A^M) - F(A^*) \| \leqslant \| f - F(A^M) \| - \| f - F(A^*) \|. \tag{6}$$

Also, $\| f - F(A^M) \| = | f(x_0) - F(A^M, x_0) |$, for some $x_0 \in [0, 1]$ and there exists $x_M \in M$ such that $| x_M - x_0 | < \delta$. From (ii) in Section 4 and Theorem 1 we see that $F(A^M) \in L$.

Thus,

$$\begin{aligned} \|f - F(A^M)\| &= |f(x_0) - F(A^M, x_0)|, \\ &\leqslant |f(x_0) - f(x_M)| + |f(x_M) - F(A^M, x_M)| \\ &\quad + |F(A^M, x_M) - F(A^M, x_0)|, \\ &\leqslant \omega(\delta) + \|f - F(A^M)\|_M + \Omega(\delta), \\ &\leqslant \omega(\delta) + \Omega(\delta) + \|f - F(A^*)\|. \end{aligned} \tag{7}$$

By (6) and (7) we have

$$\alpha \| F(A^M) - F(A^*)\| \leqslant \omega(\delta) + \Omega(\delta).$$

Thus, the conclusion holds.

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